ANALYSIS OF LOW-AMPLITUDE VIBRATION DATA

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ABSTRACT:

Analysis of low-amplitude ambient vibration data is becoming an important subject as more and more real-time ground and structural monitoring networks are being installed. Although the main objective on these networks is to monitor earthquakes, most of the recorded continuous data are of ambient vibrations. Ambient vibration records are characterized by their low amplitudes and low signal-to-noise ratios (SNRs). Fourier-based spectral analysis methods that are commonly used to analyze vibration records are not always appropriate to analyze ambient data. Low signal-to-noise ratios cause large errors in spectral analysis, particularly when analysis involves spectral ratios. Typically, ambient ground vibration records are stationary, linear, and can be made infinitely long. These properties allow utilization of statistical signal processing techniques for data analysis, which are much superior to spectral analysis when dealing with noisy data. The paper presents some simple techniques to reduce the effects of noise in spectral analysis. By using elementary concepts from the statistical signal processing and optimal filtering theories, the paper introduces several methods to analyze noisy ambient vibration signals. The methods include those based on the autocorrelation function and autocorrelation matrix of the record. The superiority of the new methods over the spectral methods is demonstrated by examples.

KEYWORDS : Ambient vibration analysis, ground noise, low amplitude, signal analysis.

1. INTRODUCTION

With the rapid increase in the number of real-time, continuous monitoring systems for ground and structures, it is becoming more important that Earthquake and Structural Engineers become familiar with tools and techniques to analyze low amplitude, low SNR (signal-to-noise ratio) vibration records. Such vibrations, commonly called ground noise or ambient vibrations, are generated by various sources that exist in a typical urban environment., such as wind loads and structures, forces transmitted by heavy machinery operating on the surface, vibrations generated by traffic loads, and microtremors.

Although it is commonly referred as noise (i.e., a broadband signal with no dominant frequencies), an ambient vibration record generally includes periodic components (i.e., sinusoids) buried in noise. Some of these components represent the natural frequencies of soil layers near the surface, or the natural frequencies of the building. Analysis of such signals involves extracting these sinusoids from the records.

In general, signal-to-noise ratios (SNR) in ambient vibration records are very low, typically less than one. The standard Fourier-based spectral analysis techniques may not always work for ambient data. The low SNR in the records cause large errors in spectral analysis, particularly when analysis involves spectral ratios. However, ambient vibrations also present some advantages for data analysis. The data are always available and the record length can be made infinitely long. The signals are stationary, which means that the temporal and frequency characteristics of the signal do not change with time. Also, since there are a large number of sources for noise and excitation, it can reasonable be assumed that both the noise and the excitation are wide-band random processes. These properties of ambient data make it possible to utilize advance statistical signal processing techniques for analysis.

2. MINIMIZING EFFECTS OF NOISE

Fourier-based spectral analysis has been the standard method to analyze vibration records. The main source of errors in spectral analysis is the noise in the records. Noise alters the amplitudes and frequency content of Fourier spectra, and introduces spurious resonant peaks. Taking spectral ratios can magnify the errors to unacceptably high levels. Three techniques are described below to improve the accuracy of Fourier spectra.

2.1 Segmentation and averaging

Assume that the recorded signal, *x(t),* is the sum of actual (i.e., noise free) signal, *s(t),* plus zero-mean Gaussian white noise, *n(t)*:

$$x(t) = s(t) + n(t) \quad \text{with } n(t) = \mathrm{N}\left[0,\sigma\right] \tag{1}$$

The discrete-time Fourier spectra of noise-free and noisy signals are

$$s(t) = \sum_{k=1}^{N/2+1} a_k \cdot \cos(2\pi f_k t) + \sum_{k=1}^{N/2+1} b_k \cdot \sin(2\pi f_k t)$$
$$x(t) = \sum_{k=1}^{N/2+1} \hat{a}_k \cdot \cos(2\pi f_k t) + \sum_{k=1}^{N/2+1} \hat{b}_k \cdot \sin(2\pi f_k t) \tag{2}$$

For zero-mean Gaussian *n(t)*, we can calculate the statistical properties of the Fourier coefficients of *x(t)*, and show the following:

$$\text{Mean}[\hat{a}_k] = a_k \quad \text{and} \quad \text{Mean}[\hat{b}_k] = b_k$$
$$\text{Variance}[\hat{a}_k] = \frac{2\sigma^2}{N} \quad \text{and} \quad \text{Variance}[\hat{b}_k] = \frac{2\sigma^2}{N} \tag{3}$$

where σ^2 is the variance of *n(t)* and *N* denotes the number of points in the record. The first equation confirms that the mean values of the Fourier coefficients of the noisy signal, *x(t),* are equal to those of the noise-free signal, *s(t).* The second equation shows that the variances of the Fourier coefficients of *x(t)* are inversely proportional to the record length; that is, the longer the record length the smaller the variance of Fourier spectrum (i.e., the more accurate the results). This observation suggests that we should consider very long signals when calculating the Fourier spectra of ambient data, provided that the signal characteristics remain stationary. If the stationarity condition is not met, the alternative would be to divide the signal into equal-length stationary segments, and calculate the Fourier spectrum as the average of the Fourier spectra of these segments.

2.2 Selection of smoothing windows

A widely used technique to reduce the influence of noise in Fourier spectra is to apply smoothing windows. There are no straightforward rules on selecting smoothing windows. Too short smoothing windows may not provide sufficient noise reduction, whereas too long smoothing windows may eliminate some of the real peaks, as shown by example in Fig. 1. A simple technique for selecting the optimal smoothing window length is suggested in (Safak, 1997). It involves plotting the area under the squared Fourier amplitude spectrum with increasing window length. The plot shows a decaying curve with increasing window length. Initially, the decay is very fast, but becomes much slower as the window length increases. If it is assumed that the noise-free Fourier amplitude spectrum is a smooth function of frequency, it can be shown that the window length, where the rate of decay in the curve changes from fast to slow, corresponds to the optimal window length. The procedure for finding this point in the curve is shown schematically in Fig. 2.

When calculating site amplification by spectral ratios (e.g., soil-to-rock ratios or H/V ratios) a common practice is to use the same smoothing windows for the numerator and denominator spectra. There is no scientific justification for using the same smoothing windows. In general, the SNR of soil records are much higher than those of rock records. Similarly, the SNR of horizontal records are higher than those of vertical records. Therefore, when taking the spectral ratios, the numerator and denominator Fourier spectra should be smoothed independently by their corresponding optimal smoothing windows.

2.3 Least-squares estimation of Fourier spectra

The discrete Fourier expansion of the noise-free signal, *s(t)*, is given by the first expression in Eq. 2. For a given signal length, *N*, and sampling interval, Δt, the discrete frequencies, f_k, of the Fourier spectra are defined by the following equation

$$f_k = \frac{k}{N \cdot \Delta t} \quad \text{where} \quad k = 1, \cdots, (N/2+1) \tag{4}$$

Therefore, all the sine and cosine terms in the Fourier expansion of *s(t)* are known. The unknowns are the Fourier coefficients, a_k and b_k. Instead of determining a_k and b_k by standard Fast Fourier transforms, we can calculate them by minimizing the error, *V*, between the noise-free signal, *s(t)*, and the recorded signal, *x(t)*, using the following equations:

$$V = \sum_{t=1}^{N} \left[x(t) - s(t) \right]^2 \quad \text{where} \quad s(t) = \sum_{k=1}^{N/2+1} a_k \cdot \cos(2\pi f_k t) + \sum_{k=1}^{N/2+1} b_k \cdot \sin(2\pi f_k t)$$
$$\min_{a_k, b_k}(V) \quad \rightarrow \quad \frac{\partial V}{\partial a_k} = 0 \quad \text{and} \quad \frac{\partial V}{\partial b_k} = 0 \tag{5}$$

The minimization results in a linear set of equations for a_k and b_k, which can easily be solved by matrix inversion. The calculated a_k and b_k represent the least-squares estimate of the Fourier coefficients of the noise-free signal. Figure 3 shows a low-amplitude vibration signal taken from the first story of a 10-story building and the corresponding standard and least-square Fourier amplitude spectra. The least-squares Fourier spectra have smaller amplitudes outside the dominant frequency band (i.e., 1.0 to 2.0 Hz). The reduction is due to the minimization of noise amplitudes in those regions.

3. AUTOCORRELATION FUNCTIONS AND OPTIMAL FILTERING

The auto-correlation function *R(τ)* of a signal *x(t)* is defined by the following equation:

$$R(\tau) = \frac{1}{N} \sum_{t=1}^{N} x(t) \cdot x(t \quad \tau) \tag{6}$$

For stationary signals, such as ambient ground noise, the auto-correlation function depends only on the time lag τ. It can be shown that the expected autocorrelation function of a sinusoid buried in noise has the same frequency as the sinusoid. That is

$$\text{If} \quad x(t) = A \cdot \cos(\omega t) + n(t) \quad \rightarrow \quad \mathrm{E}[R(\tau)] = \frac{A^2}{2} \cdot \cos(\omega t) \tag{7}$$

where E[ ] denotes the expected value. In other words, taking the autocorrelation does not change the frequency content of the signal. It can also be shown that the autocorrelation improves the SNR, i.e., the SNR in the autocorrelation of a signal is higher than that of the original signal. This is because the autocorrelation operation amplifies the amplitudes of any periodic components in the data. Therefore, when calculating Fourier spectra of ambient noise, it is advantageous to use the autocorrelation functions of the records instead of the original records. The Fourier spectrum of the autocorrelation function is commonly known as the power spectral density function.

A concept directly related to autocorrelation functions is the optimal filtering. Optimal filtering aims to remove noise by searching correlated (i.e., periodic) components in the record. We assume that the periodic components in the record correspond to the actual (i.e., noise free) signal, and the remaining components are considered to be the noise. A characteristic of a periodic signal is that its value at any given time can be written as a linear combination of its past values. Therefore, if we were able to separate the record, *x(t),* into its periodic and random components, we can expres *x(t)* as

$$x(t) = \sum_{k=1}^{m} a_k \cdot x(t-k) + n(t) \tag{8}$$

where the first term on the right hand side is the periodic component. Assume that we know all the values of *x(t)* up to time step *(t-1)* and want to predict the value at the next time step, *t*. Since the mean value of *n(t)* is zero, the most likely value, $\hat{x}(t)$, of *x(t)* would be

$$\hat{x}(t) = \sum_{k=1}^{m} a_k \cdot x(t-k) \qquad (9)$$

The difference between the predicted and the recorded values of *x(t)* is the error in our estimation. We can select the coefficients a_k in Eq. 9 such that the estimation error, *V*, is minimum. That is,

$$\min_a(V) = \left[x(t) - \sum_{k=1}^{m} a_k \cdot x(t-k) \right]^2 \quad \rightarrow \quad \frac{\partial V}{\partial a_k} = 0 \qquad (10)$$

Eq. 10 results in a set of linear equations to determine the coefficients a_k. These coefficients define the filter to remove noise from the signal. We calculate the noise-free signal by filtering the record as shown in Eq. 9.

The procedure presented above describes the basic idea in optimal filtering. There are numerous variations of the procedure suggested in the literature with their unique names such as Wiener filtering, Recursive Least Squares, Least Mean Squares, Durbin Algorithm, Burg Algorithm, and Yule-Walker Algorithm. More detail on these methods can be found in textbooks on optimal filtering and linear estimation (e.g., Kailath et al., 2000).

4. EIGENVALUES OF AUTOCORRELATION MATRIX

Another set of powerful tools to separate signal from the noise can be developed based on the eigenvalues and eigenvectors of the autocorrelation matrix. The autocorrelation matrix, *Q*, is defined by the following equation:

$$Q = \begin{pmatrix} R(0) & \dots & R(M) \\ \vdots & \ddots & \vdots \\ R(-M) & \cdots & R(0) \end{pmatrix}$$

where (11)

$$R(\tau) = \frac{1}{N} \sum_{t=1}^{N} x(t) \cdot x(t-\tau) \quad \text{and} \quad \tau = -M, \cdots, 0, \cdots, M;$$

Q is a $(M+1)\times(M+1)$ dimensional matrix that has *(M+1)* eigenvalues and eigenvectors. The well known Karhunen-Loeve expansion states that a stationary signal can be represented in terms of the eigenvectors of its autocorrelation matrix. That is

$$x(t) = \sum_{i=0}^{M} c_i \cdot q_i(t) \tag{12}$$

where $q_i(t)$ denotes the *i*'th eigenvector and c_i is a constant. It can also be shown that the eigenvalues that correspond to the correlated (i.e., periodic) components of the record are much larger than those that correspond to the uncorrelated (i.e., noise) components in the record. Therefore, the eigenvalues and eigenvectors of the correlation matrix can be used to separate the noise from the signal.

There are several filtering methods based on this approach, such as Pisarenko Harmonic Decomposition, Multiple Signal Classification (MUSIC), and Rotational Invariance Techniques (ESPRIT). Details of these methods can be found in advanced textbooks on signal processing (e.g., Moon and Stirling, 2000). Figure 4 shows a low-amplitude vibration record and the corresponding standard, auto-correlation based (Burg), and eigen-based (MUSIC) Fourier amplitude spectra. There is a significant reduction in noise effects by the Burg and MUSIC algorithms.

5. CONCLUSIONS

Ambient ground noise and ambient vibrations of structures are generated by small excitation sources typically present in an urban environment. In general, such vibrations include low-amplitude sinusoidal components representing the resonant frequencies of the soil or the structure. Analysis of these records involves extracting those sinusoids that are buried in noise.

The standard spectral analysis methods are not always appropriate to analyze ambient data because of low signal-to-noise ratios, particularly when analysis involves spectral ratios. The influence of noise in spectral analysis can be reduced by considering longer records, or by averaging the spectral content of a number of shorter records. Using smoothing windows with optimal length or calculating Fourier coefficients by the least- squares approximation also improves the accuracy of spectral analysis. The stationarity of ambient records allow development of advanced filtering techniques, based on the properties of the autocorrelation function and the eigenvalues of the autocorrelation matrix.

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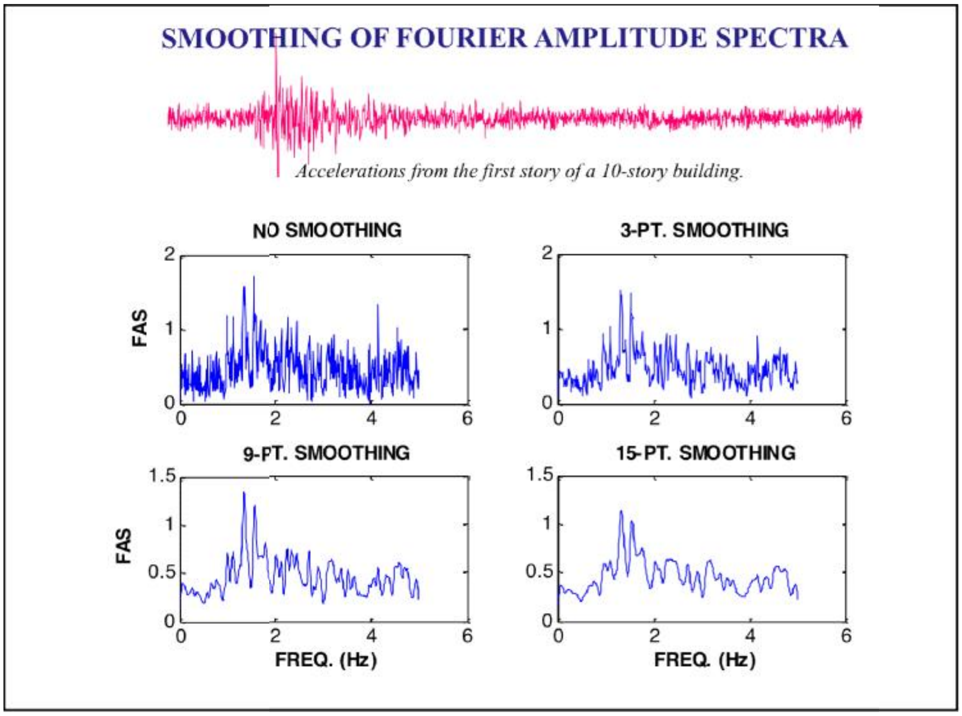

Figure 1. Effect of smoothing windows on Fourier Amplitude Spectrum.

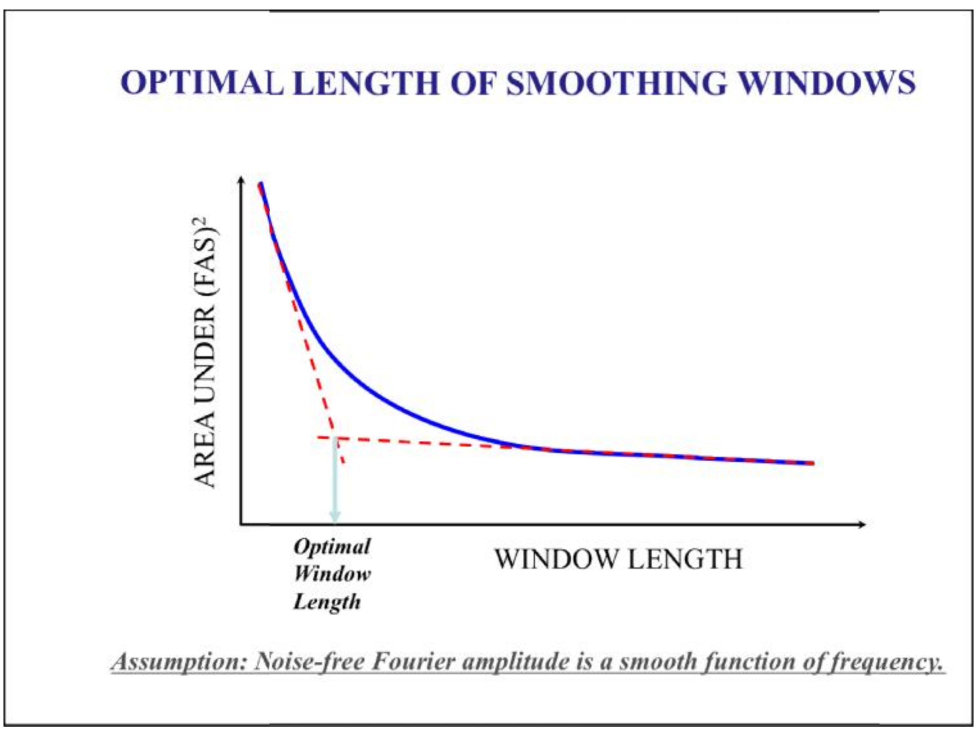

Figure 2. Selection of optimal smoothing window for Fourier Amplitude Spectrum.

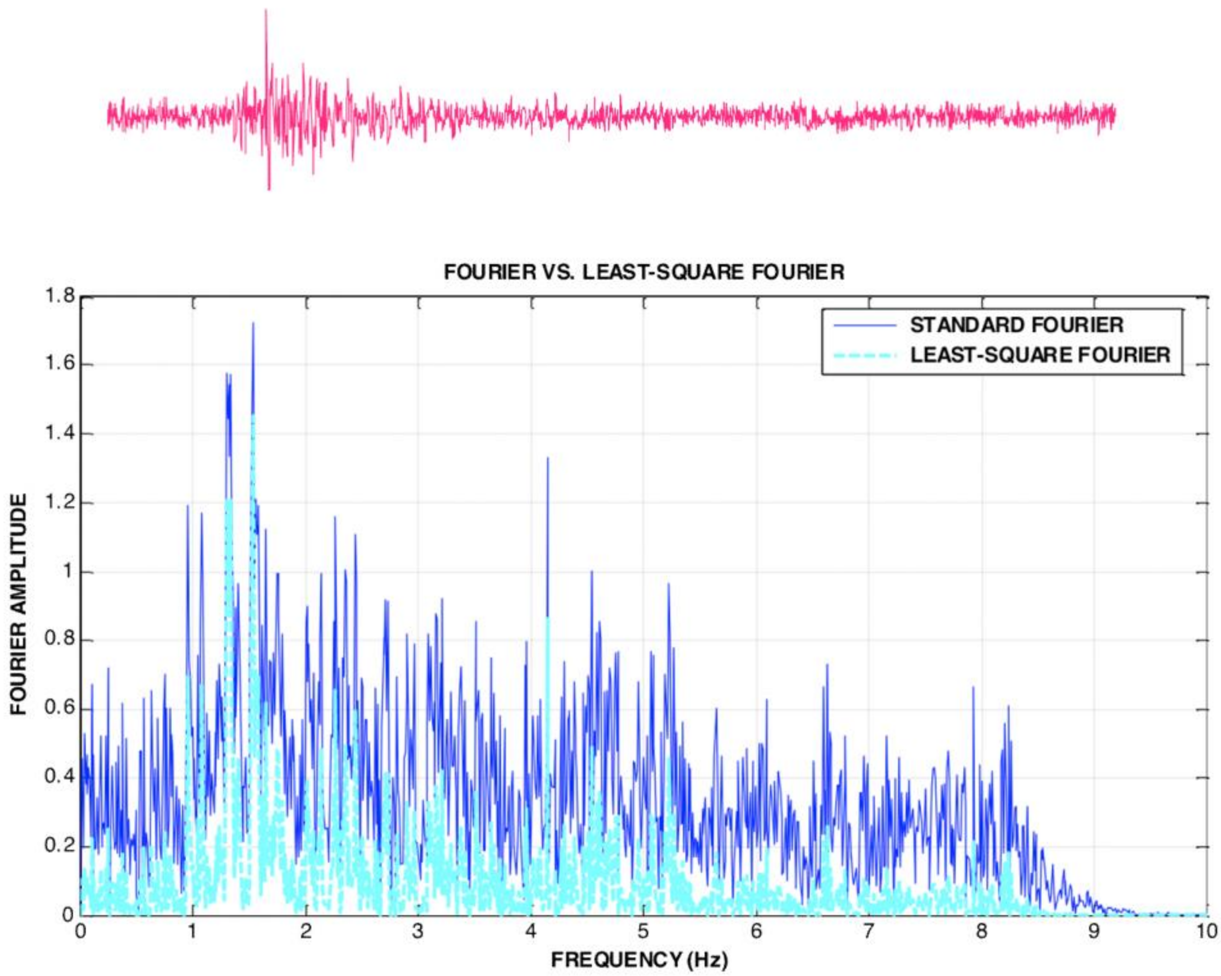

Figure 3. Comparison of standard versus Least-Squares Fourier Amplitude Spectra.

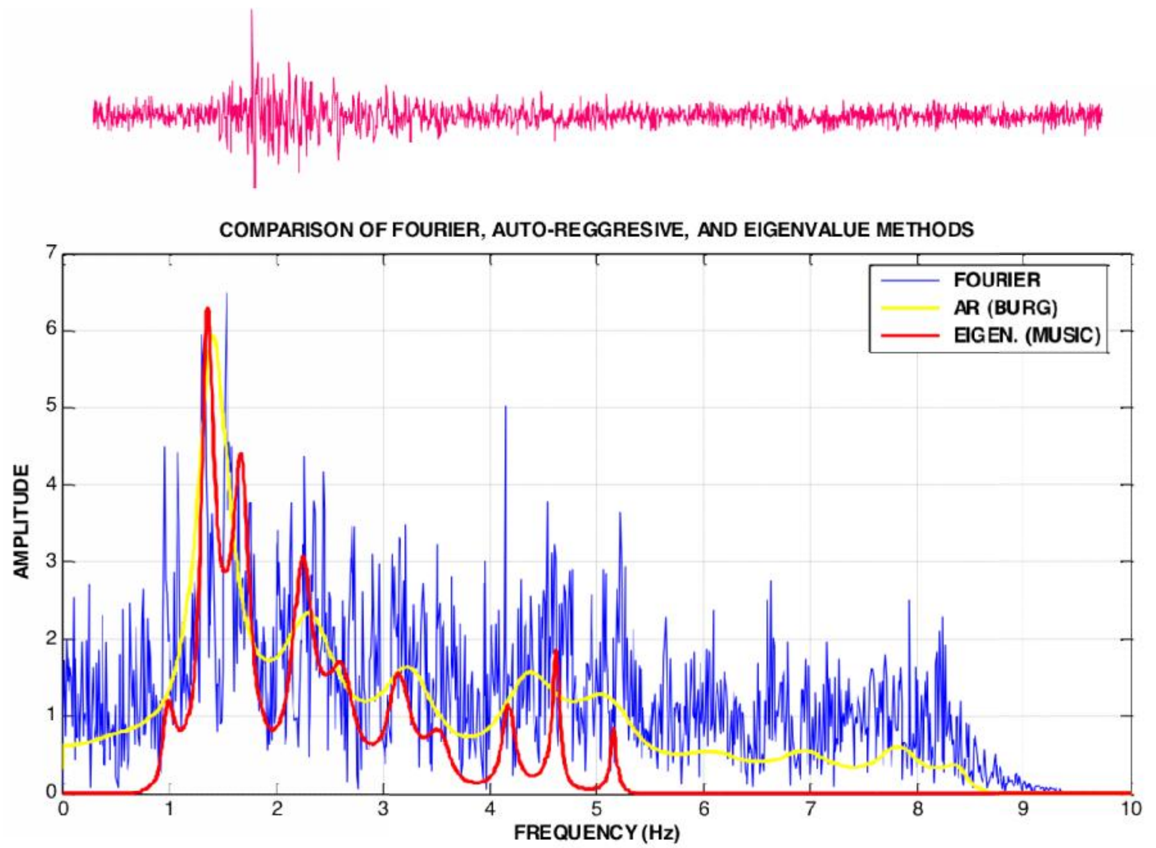

Figure 4. Comparison of Fourier Amplitude Spectra calculated by using standard Fourier analysis, auto-correlation approach (Burg method), and eigenvalue approach (Music method).