THE INTERACTION OF ELASTICITY AND ROCKING IN FLEXIBLE STRUCTURES ALLOWED TO UPLIFT

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ABSTRACT:

Numerous structures uplift under the influence of strong ground motion. While the effects of base uplift on very stiff structures which can be idealized as rigid bodies are well-understood, relatively few studies investigate the rocking response of flexible structures. Related practical analysis and design methods treat these structures with simplified 'equivalent' oscillators. This paper addresses the fundamental dynamics of flexible rocking structures. The nonlinear equations of motion for the idealized structural model were derived using a Lagrangian formulation for large rotations. Particular attention was devoted to the transition in between successive phases; a physically consistent classical impact framework was utilized alongside an energy approach. The fundamental dynamic properties of this system were compared with those of linear elastic oscillators and rigid rocking structures. Results have revealed the distinct characteristics of flexible rocking structures which arise from the complex interaction of elasticity and rocking. Preliminary results on the ramifications of this interaction on the failure of flexible rocking structures are presented.

KEYWORDS: rocking, overturning, resonance, analytical dynamics, earthquake engineering

1 INTRODUCTION

The investigation of rocking structures was initially motivated by the surprising stability of flexible and slender structures under earthquake excitations. In his seminal paper, Housner (1963) cited the uplifting and the subsequent rocking motion as a reason for their stability. His work formulates the rocking motion with a rigid block model which effectively rules out the failure of the structure through strength deficiency and emphasizes the overturning instability. A plethora of studies which followed investigated the complex dynamics that govern the overturning response of the rigid block. In particular, Makris and Konstantinidis (2003) drew attention to the distinct fundamental dynamics of rocking which depart significantly from those of linear elastic oscillators.

Relatively few studies have investigated rocking structures with flexible models and their approach emphasized strength considerations. Meek (1975) challenged the assumption that structures stick to the ground during earthquakes and investigated the dynamic response of a flexible lumped mass structure allowed to uplift. This study and further work in the field of flexible rocking structures (e.g. Oliveto et al. 2003) devote little attention to rocking amplitude or overturning instability and focus on how uplift affects the structural deformations. In some cases, equivalent oscillators are proposed to estimate the maximum structural deformation experienced in the uplifted stage (e.g. Chopra and Yim 1985). Research into rocking unreinforced masonry walls (Lam et al. 2003) and modern bridge piers (Palermo et al. 2004) adopt a similar approach which disregards the interaction of elasticity and rocking when defining simplified load deformation curves.

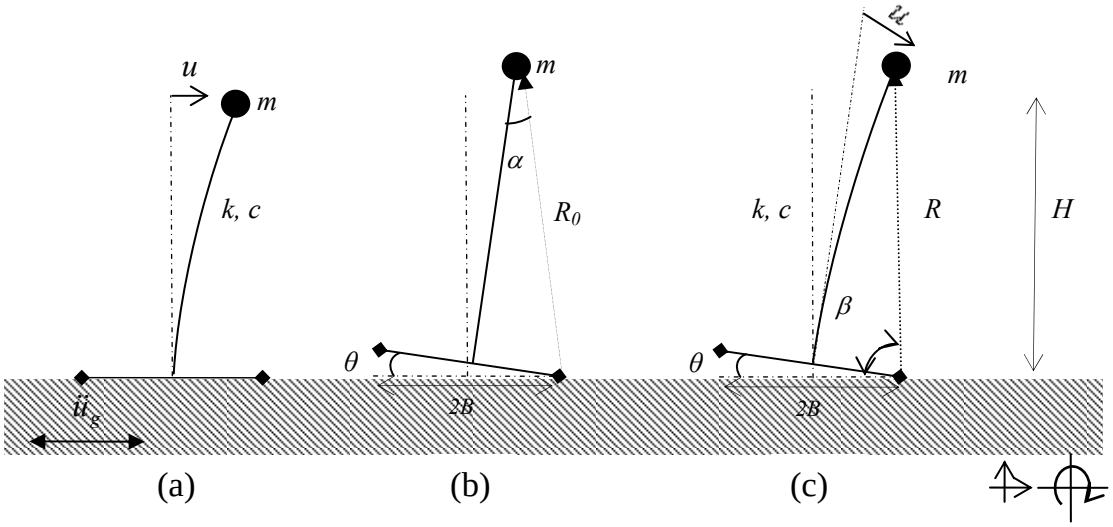

Figure 1 - Structural models and parameters considered in the analyses - (a) Linear Elastic Oscillator, (b) Rigid Rocking Structure, (c) Flexible Rocking Structure

This research investigates the interaction of elasticity and rocking in an attempt to clarify the fundamental behavior of flexible rocking systems. In particular, the aim is to quantify the effects of this interaction on the overturning (stability) and resonance (strength) of flexible rocking structures and to evaluate the use of equivalent systems in analyses and design. Specifically, the ability of rigid systems to predict the overturning instability of flexible structures, and the ability of equivalent oscillators to determine the structural deformations of the flexible rocking systems was investigated.

2 ANALYTICAL MODELS

Idealized models were used to represent three typologies of generic structures: linear elastic oscillators, rigid rocking structures and flexible rocking structures (see Figure 1). These models assume that the ground is rigid and that it has a very high static coefficient of friction so that no sliding occurs between the base and the foundation. Furthermore the point masses in structures Figure 1b and Figure 1c have negligible moment of inertia.

Figure 1 shows the structures and the parameters involved. The linear elastic oscillator (Figure 1a) can exhibit elastic translational motion only, the rigid rocking structure (Figure 1b) may only rotate about either of its corners. The flexible rocking structure (Figure 1c) exhibits both of these actions. The parameter u is the elastic translation of the mass and θ is the rigid body rotation of the foundation. Alternatively, the response can be defined by R, the distance of the lumped mass from the base pivot, and β, the Lagrangian rotation parameter. The classical set of parameters (u,θ) was used in the representation and evaluation of results whereas the Lagrangian set of parameters (R,β) was used in the derivation of the equations of motion and the transition of phases.

The equations of motion of linear elastic oscillators and rigid rocking structures are well documented in the literature. This work presents the equations of motion of the two phases of the motion of flexible rocking structures: (i) the full contact phase and (ii) the rocking phase. In the full contact phase, the flexible rocking structure behaves like a linear elastic oscillator. Its equation of motion is given by:

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g \tag{1}$$

where $\ddot{u}_g$ is the horizontal ground motion, $\omega_n = \sqrt{k/m}$, is the natural frequency of the system and $\zeta = c/(2\sqrt{km})$ is the damping factor. A flexible structure with quiescent initial conditions, initially responds elastically until uplift is realized when the overturning moment exceeds the resisting moment due to gravity. This condition is checked at each time step and is described by:

$$\left|mH(\ddot{u} + \ddot{u}_g)\right| > \left|mg(B \mp u)\right| \tag{2}$$

where the upper sign represents rocking condition about the right base corner and the lower sign about the left base corner. Similar sign notation will be used throughout this paper to represent rocking in each direction. Upon uplift, the rocking phase begins. The equations of motion for the rocking phase utilize (R, β) and are derived for large rotations and small elastic deformations:

$$\begin{aligned} \ddot{R} &= \omega_n^2 R\left(\frac{B}{\sqrt{R^2 - H^2}} - 1\right) - \frac{2\zeta\omega_n R^2}{R^2 - H^2}\dot{R} + R\dot{\beta}^2 \pm \ddot{u}_g \cos\beta \mp g\sin\beta \\ \ddot{\beta} &= \frac{-2\dot{R}\dot{\beta}}{R} \mp p^2\left(\cos\beta \mp \frac{\ddot{u}_g}{g}\sin\beta\right) \end{aligned} \tag{3}$$

where the frequency parameter $p = \sqrt{g/R}$ and g denotes the gravity. These equations are highly nonlinear, contain coupling between parameters and are piecewise defined. Different equation sets are valid for $\theta > 0$ and $\theta < 0$, thus at each time step, parameters (R, β) are converted to (u, θ). When $\theta \approx 0$, the integration was stopped and contact conditions were assessed to determine the next phase of motion. To do this, fictitious full contact and rocking phases were defined and the kinetic energy in each was compared to determine the next phase (Oliveto et al. 2003). The fictitious full contact phase occurs when upon impact the body assumes full contact and continues its motion deforming elastically. The vertical momentum is fully dissipated and the conservation of linear momentum dictates that:

$$\dot{u}_2 = \dot{u}_1 + H\dot{\theta}_1 \tag{4}$$

where $\dot{u}_2$ is the post impact full contact fictitious velocity and the subscript 1 indicates pre-impact parameters. The impact is instantaneous and the position of the mass does not change during impact. After impact $u_2 = u_1$ and $\theta_2 = \dot{\theta}_2 = 0$. The kinetic energy of the fictitious full contact phase is thus given by:

$$E_{fc} = \frac{1}{2}m\dot{u}_2^2 \tag{5}$$

The post-impact parameters of the fictitious rocking phase were derived with the use of a classical impact framework where the effects of elasticity on post-impact parameters were also considered, yielding:

$$\dot{\beta}_2 = \left(\frac{R_1^2 + H^2 - (B-u)(B+u)}{2R_2^2}\right)\dot{\beta}_1 \mp \left(\frac{BH}{R_1 R_2^2}\right)\dot{R}_1 \tag{6}$$

$$\dot{R}_2 = \dot{R}_1 \tag{7}$$

Equations 6 and 7 were used in conjunction with two constraints: the kinetic energy of the post-impact state must be less than that of pre-impact and no bouncing is allowed. The kinetic energy of the fictitious rocking phase is given by the following expression:

$$E_r = \frac{1}{2} m(R^2 \dot{\beta}^2 + \dot{R}^2) \tag{8}$$

To determine the phase of motion after impact, Equations 5 and 8 were compared. If $E_r > E_{fc}$, a full contact phase follows impact with initial conditions set by Equation 4. If $E_{fc} > E_r$, a rocking phase is initiated with post-impact parameters given by Equations 6 and 7. These equations of motion and criteria for phase transition describe the dynamic motion of the flexible rocking structure shown in Figure 1c.

3 DIMENSIONLESS SYSTEM AND GROUND MOTION PARAMETERS

Two types of periodic excitation will be utilized in the analyses: trigonometric pulses and harmonic excitations. Pulse excitations will be instrumental in understanding the effects of elasticity on overturning instability as pulses have been reported in the literature to be the driving force of overturning collapse of rigid rocking structures (Zhang and Makris, 2001). Harmonic excitation will be used to determine the effects of resonance on flexible rocking structures and investigate if any single frequency excitation can drive the structure to failure, either through excessive deformations or overturning instability.

Due to the nonlinearity of the coupled equations of motion of the rocking phase, numerical integration is required. One of the challenges in the analysis of nonlinear systems is to present the results in an intuitive and informative manner for a range of different structures. This study employs formal dimensional analysis for this purpose. Six fundamental dimensionless input parameters are required to describe the system. Three response parameters are defined and are given as follows:

$$\pi_u = \frac{u_{\max}\omega_n^{\,2}}{g\alpha} = \frac{u_{\max}}{u_{cr}},\ \pi_\theta = \theta, \pi_E = \frac{E}{mgR_0(1-\cos\alpha)} = \frac{E}{E_r} \tag{9}$$

where $u_{\max}$ is the maximum elastic displacement that the flexible rocking structure experiences under the given base excitation , u_{cr} is the critical displacement at which uplift occurs for an undamped structure, E is the total energy of the system and E_r is the potential energy of the rigid rocking block at its unstable equilibrium position, $\theta = \alpha$. The input parameters are given by:

$$\pi_n = \frac{\omega_n}{p}, \pi_w = \frac{\omega}{p},\ \pi_A = \frac{A}{g\alpha}, \pi_\alpha = \alpha,\ \pi_\zeta = \zeta,\ \pi_t = pt = \tau \tag{10}$$

where ω is the frequency and A is the amplitude of the periodic excitation. The chosen set of dimensionless parameters yield a physically similar response under the influence of pulse-type and harmonic ground excitations. This allows the representation of generalized results which are valid for a wide class of flexible rocking structures subjected to different periodic base excitations.

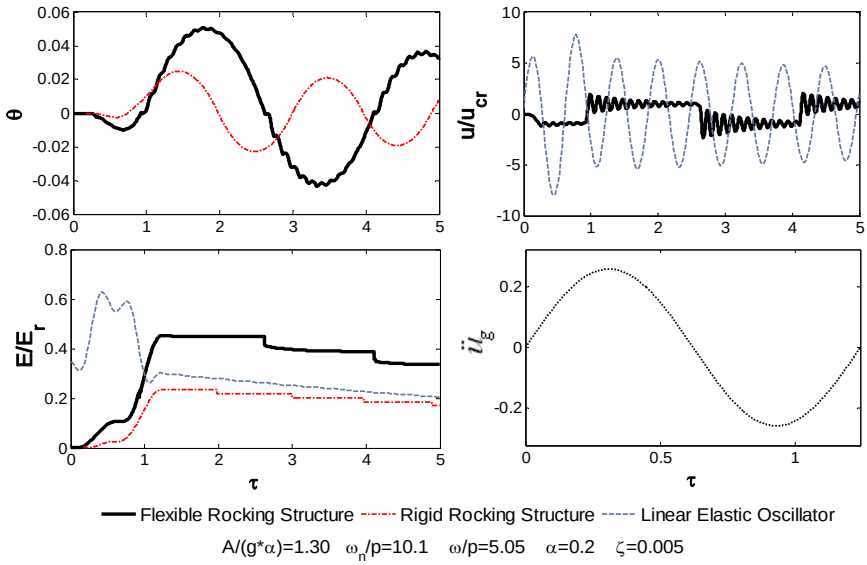

Figure 2- The rocking, dimensionless deformation and the dimensionless energy response of similar systems and the sine pulse ground motion utilized in the analyses

4 COMPARISON OF FUNDAMENTAL DYNAMICS

To highlight the differences between the systems shown in Figure 1, structures with similar properties were subjected to the same sine pulse excitation. The flexible rocking structure was specified to have the same natural frequency and damping as the similar linear elastic oscillator and the same geometric properties as the similar rigid rocking structure. Figure 2 shows the response of these similar structures.

Significant differences can be observed in the rocking response of the flexible rocking structure. A high frequency oscillation in the rocking curves indicates the coupling of elasticity and rocking. This oscillation is the strongest at the beginning of each rocking cycle and as the elastic motion is damped out, the rocking action of the flexible structure dominates the motion. Another important difference between the rigid rocking system and the flexible rocking system relates to the transition in between rocking cycles. While the rigid structure proceeds directly from one rocking cycle to another, accompanied by a loss in angular velocity, the transitions of flexible structures are more complicated. In the specific case of Figure 2, upon every impact the structure proceeded to a new rocking cycle directly. However, the elasticity effects counteract the rocking action and cause the rocking cycle just after the first impact to be very short in amplitude and duration. A short full contact phase follows and upon reaching the uplift condition, rocking about the same corner takes place. In short, the elasticity effects counteract rocking action resulting in a rotation response significantly different in shape, amplitude and duration.

The structural deformations of the flexible rocking system are markedly different from the similar linear elastic oscillator. The natural frequency and damping of the uplifted system is higher. The deformation response can be conceived as the sum of an average distortion and a high frequency oscillation with heavy damping. The elastic motion of the uplifted state seems to be independent of the ground motion as oscillations are fully damped out despite continuing ground motion. However, the observation that the dimensionless deformation response in the first rocking cycle differs significantly from the rest shows the sensitivity of the deformation response to initial conditions of the rocking cycle. These fundamental dynamic differences are summarized in Table 1 where $\omega_{n,u}$ and ζ_u denote the uplifted natural frequency and damping respectively.

Table 1 The comparison of fundamental properties of three systems shown in Figure 1

Parameters	***Linear Elastic Oscillator***	***Rocking Rigid Structure***	***Rocking Flexible Structure***
Restoring Mechanism	Elasticity	Gravity	Gravity and Counteracting Elasticity
Restoring Force/Moment	$F_r = ku$	$M_r = mgR\sin(\alpha-\theta)$	$M_r = mgR_0\sin(\alpha-\theta) \pm m\ddot{u}H$
Stable Equilibrium	$u = 0$	$\theta = 0$	$u = 0$
Stiffness At Stable Equilibrium	Finite	Infinite	Finite
Stiffness Away From Stable Equilibrium	Positive	Negative	Dependent upon phase of motion.
Frequency Parameter(s)	$\omega_n = \sqrt{\frac{k}{m}}$	$p = \sqrt{\frac{g}{R}}$	$\omega_n = \sqrt{\frac{k}{m}}, \omega_{n,u} \cong \frac{R}{B}\sqrt{\frac{k}{m}}, p = \sqrt{\frac{g}{R}}$
Uplift amplitude and Rocking Phase Angle	Uplift is not allowed	$A = g\tan^{-1}(\frac{B}{H})$ $\phi = \sin^{-1}(\frac{A}{g\alpha})$	Uplift and rocking phase angle are $f(\frac{\omega}{\omega_n}, \zeta, \frac{A}{g\alpha})$
Damping	Continuous, $\zeta = \frac{c}{2\sqrt{km}}$	At impacts only $\frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{H^2}{R^2}$	Continuous & at impacts $\zeta = \frac{c}{2\sqrt{km}}, \zeta_u = \frac{R}{B}\zeta$ and see Equations 4-8

5 THE EFFECTS OF INTERACTION OF ELASTICITY AND ROCKING

Section 4 outlined some of the fundamental dynamic differences between the systems shown in Figure 1 arising from the complex interaction of elasticity and rocking. Of practical interest is how these differences affect the failure of the rocking structure either through overturning (stability) or resonance (strength).

Figure 3 shows the rotation and dimensionless deformation response of similar structures to different excitations. This figure depicts the differing overturning response of flexible and rigid rocking structures. In the upper row of Figure 3 the flexible structure overturns under a low frequency excitation which does not even uplift the rigid structure. This emphasizes the role of uplift in the overturning response of rocking structures. In the lower row of Figure 3 the same flexible structure survives a different pulse and starts to rock freely, while the rigid structure overturns. It can be observed that the counteracting elasticity effects cause multiple impacts after each rocking cycle dissipating more energy and preventing the failure of the flexible rocking structure. These graphs point out significant differences in the overturning instability of flexible rocking structures that requires investigation.

Psycharis (1991) has shown that exciting a flexible rocking structure at its natural frequency generates limited deformations and does not cause the build-up of rocking motion. Another possibility to facilitate resonance is to

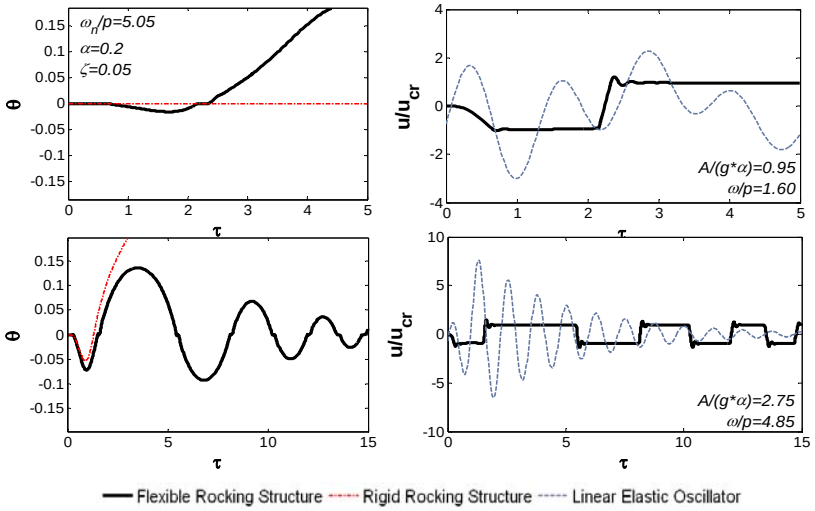

Figure 3 - The rocking and dimensionless elastic deformation response of similar systems to different frequency and amplitude sine pulse excitations.

excite the structure at its uplifted resonant frequency $\omega_{n,u}$ (see Table 1) with harmonic excitation. Figure 4 shows such a case. The high frequency excitation causes oscillations in the rocking response which, due to coupling, excites the uplifted structure at its resonant frequency. This coupling excitation is magnified by resonance and causes excessive deformations, much higher than observed in the linear elastic response. The feedback effect of the increasing magnitude of elastic deformations on the rocking response should also be noted. The counteracting elasticity effects cause a pounding-like rocking response where very short rocking cycles reach high amplitudes of rotation. As a result, excessive deformations caused by 'uplifted resonance' could drive the structure to failure.

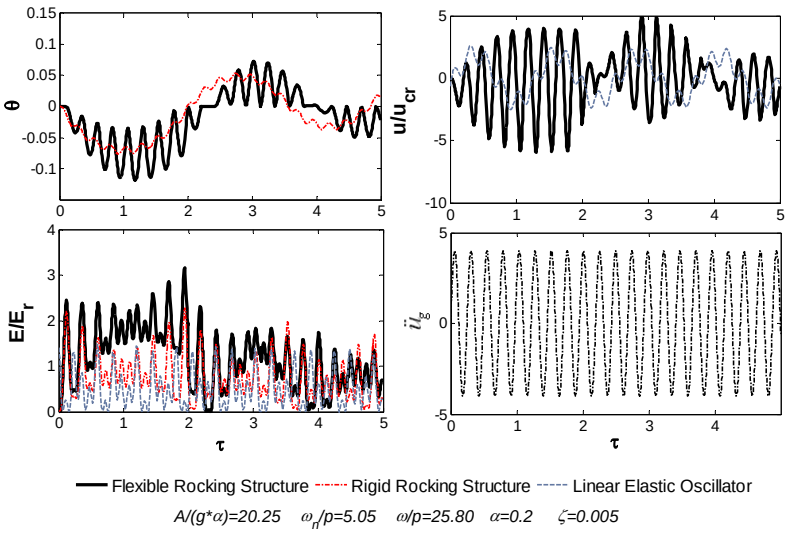

Figure 4 - The rocking, dimensionless deformation and the dimensionless energy response of similar systems to the harmonic ground motion with the uplifted natural frequency

6 CONCLUSIONS

The fundamental dynamics of linear elastic oscillators, rigid rocking structures and flexible rocking structures were compared. The differences in between these systems, arising from the complex interaction of elasticity and rocking were emphasized. The overturning response of flexible rocking structures was markedly different from its rigid counterpart and requires thorough investigation. The phenomena of uplifted resonance may result in excessive deformations which might cause failure through strength deficiency. In addition, the use of equivalent oscillators to determine structural deformations accurately requires a thorough understanding of the coupled rocking action.

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